EE 232: Lightwave Devices
Lecture #15 – Optical waveguides

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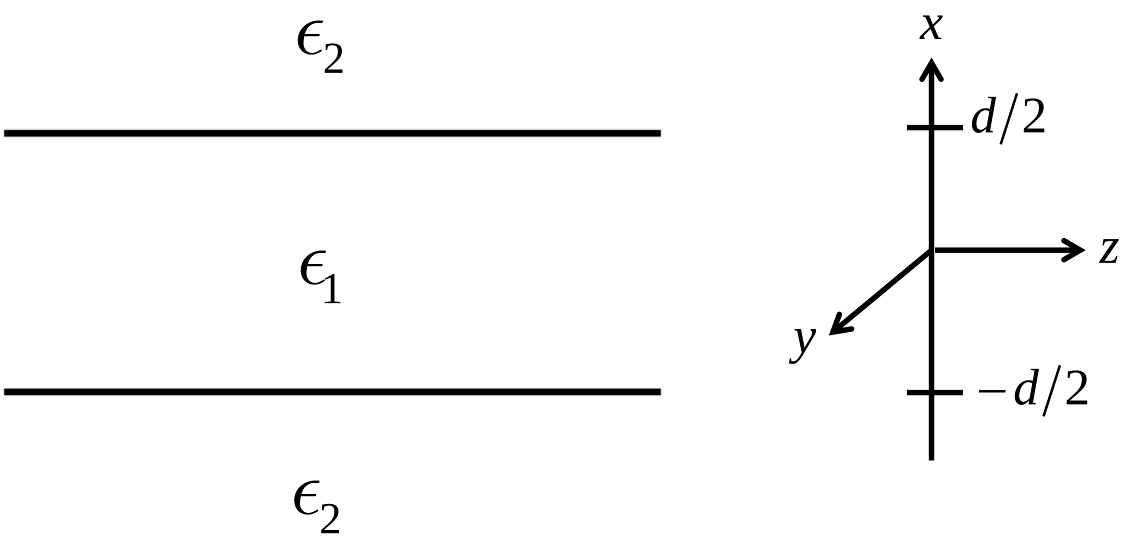
Slab waveguide



Slab waveguide consists of a slab of high-index material surrounded by low-index material ($\epsilon_1 > \epsilon_2$). The waveguide is assumed to be infinitely large in the y and z-directions.

We wish to find confined electromagnetic modes that propagate in the +z direction and solve the source-free time-harmonic wave equation

$$\left(\nabla^2 + \omega^2 \mu \epsilon\right) \mathbf{E} = 0$$

Wave equation

In general, solution for the electric-field can be written as

$$\mathbf{E}(x,y,z) = \hat{x}E_x(x,y,z) + \hat{y}E_y(x,y,z) + \hat{z}E_z(x,y,z)$$

If we plug back into the wave equation,

$$\nabla^2\mathbf{E} + \omega^2\mu\epsilon\mathbf{E} = \nabla^2\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right) + \omega^2\mu\epsilon\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)$$

We separate into three equations

$$\nabla^2 E_x + \omega^2\mu\epsilon E_x = 0 \qquad (1)$$

$$\nabla^2 E_y + \omega^2\mu\epsilon E_y = 0$$

$$\nabla^2 E_z + \omega^2\mu\epsilon E_z = 0$$

Let's expand equation (1)

$$\nabla^2 E_x + \omega^2\mu\epsilon E_x = \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0 \qquad (2)$$

Wave equation

Using separation of variables, we assume

$$E_x(x,y,z) = f(x)g(y)h(z)$$

If we plug back into equation (2)

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2$$

The sum of the terms can equal a constant only if each individual term is a constant

$$\therefore \quad \frac{1}{f}\frac{d^2 f}{dx^2} = \beta_x^2 \quad \frac{1}{g}\frac{d^2 g}{dy^2} = \beta_y^2 \quad \frac{1}{f}\frac{d^2 h}{dz^2} = \beta_z^2$$

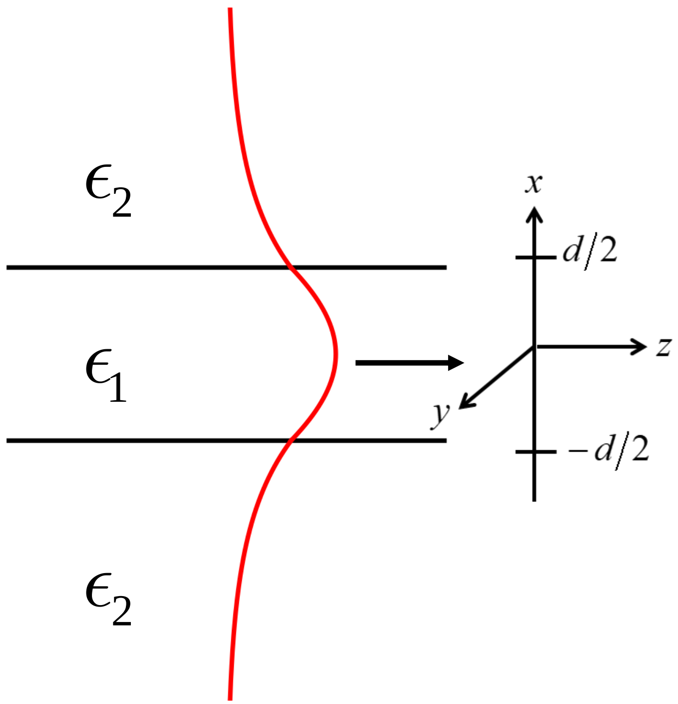
Typical solutions for these differential equations:

$$f(x) = A_1 e^{-i\beta_x x} + A_2 e^{i\beta_x x}$$ (travelling wave)

$$f(x) = B_1 \cos(\beta_x x) + B_2 \sin(\beta_x x)$$ (standing wave)

$$f(x) = C_1 e^{-i\beta_x x}$$ (evanescent wave)

Slab waveguide



We look for a solution that is transverse electric (TE), "bound" to the core of the waveguide, and travelling in the z-direction such that

$$\mathbf{E} = \hat{y}E_y(x,z) \rightarrow E_y(x,z) = f(x)h(z)$$

where we assume there is no dependence on y given the slab is translationally invariant in the y-direction.

Along the z-direction we expect a traveling wave solution

$$h(z) = C_1 e^{-i\beta_z z} + C_2 e^{i\beta_z z}$$

Along the x-direction, we expect a standing wave solution in the waveguide core and evanescent solution in the cladding.

$$f(x) = \begin{cases} A_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ A_2 \cos(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Even solution

$$f(x) = \begin{cases} B_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ B_2 \sin(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Odd solution

Slab waveguide

① Plug into wave equation →

$$\beta_x{}^2 + \beta_z^2 = \omega^2 \mu_1 \epsilon_1$$
$$-\alpha^2 + \beta_z^2 = \omega^2 \mu_2 \epsilon_2$$

② Apply boundary conditions at interface between core and cladding. Tangential component of electric and magnetic field are equal across interface. →

$$\alpha = \frac{\mu_2}{\mu_1} \beta_x \tan\left(\beta_x \frac{d}{2}\right) \text{ (even)}$$

$$\alpha = -\frac{\mu_2}{\mu_1} \beta_x \cot\left(B_x \frac{d}{2}\right) \text{ (odd)}$$

$$E_{y,core}\Big|_{x=\pm\frac{d}{2}} = E_{y,clad}\Big|_{x=\pm\frac{d}{2}}$$

$$H_{z,core}\Big|_{x=\pm\frac{d}{2}} = H_{z,clad}\Big|_{x=\pm\frac{d}{2}}$$

Slab waveguide

③ After rearranging →

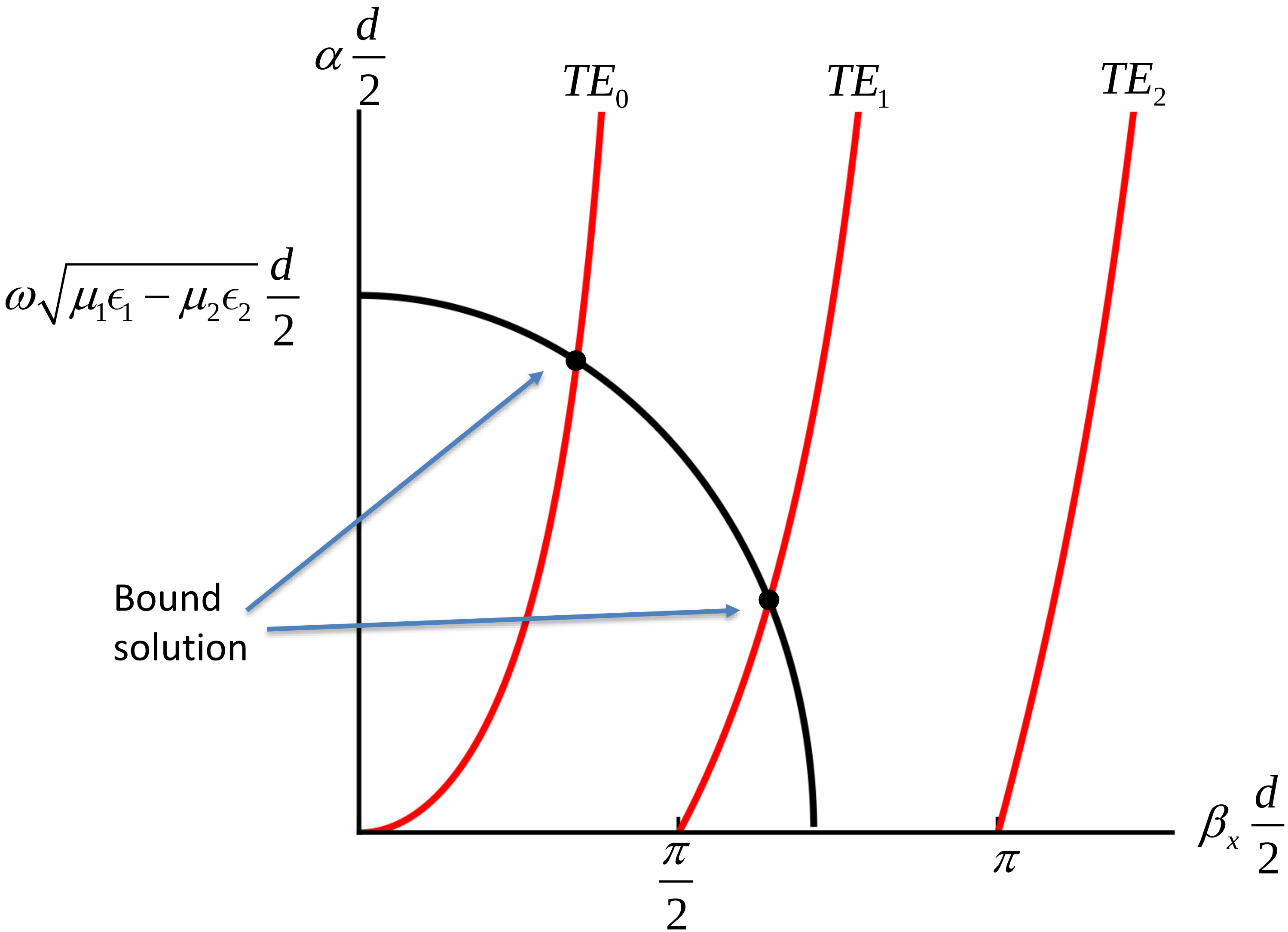
$$\left(\beta_x \frac{d}{2}\right)^2 + \left(\alpha \frac{d}{2}\right)^2 = \omega^2 \left(\mu_1 \epsilon_1 - \mu_2 \epsilon_2\right)\left(\frac{d}{2}\right)^2$$

$$\left(\alpha \frac{d}{2}\right) = \frac{\mu_2}{\mu_1}\left(\beta_x \frac{d}{2}\right) \tan\left(\beta_x \frac{d}{2}\right) \quad \text{(even)}$$

$$\left(\alpha \frac{d}{2}\right) = -\frac{\mu_2}{\mu_1}\left(\beta_x \frac{d}{2}\right) \cot\left(B_x \frac{d}{2}\right) \quad \text{(odd)}$$

Slab waveguide

④ Solve graphically



Cutoff condition

In the example on the previous slide we see that the TE_1 mode would not have a solution and would be "cutoff" if the radius of the circle is less than π/2

The cutoff condition for each mode can be generalized as

$$\omega(\mu_1\epsilon_1 - \mu_2\epsilon_2)\frac{d}{2} = m\frac{\pi}{2} \qquad m=0,1,2,3...$$

(Cutoff condition for TE_m mode)

The waveguide will be single-mode if all modes except the fundamental mode are cutoff.

$$\omega(\mu_1\epsilon_1 - \mu_2\epsilon_2)\frac{d}{2} < \frac{\pi}{2}$$

(Single mode condition)

Effective index

Effective index

$$n_{eff} = \frac{\beta_z}{\beta_0} \qquad \beta_0 = \frac{2\pi}{\lambda_0}$$

High-frequency limit

$$\text{Radius} \rightarrow \infty \;\; \text{as } \omega \rightarrow \infty$$

$$\therefore \alpha \rightarrow \infty$$

$$\begin{aligned} \beta_z^2 &= \omega^2 \mu_1 \epsilon_1 - \beta_x^2 \\ &\simeq \omega^2 \mu_1 \epsilon_1 \end{aligned} \qquad n_{eff} = \frac{\beta_z}{\beta_0} = \frac{\sqrt{\mu_1 \epsilon_1}}{\sqrt{\mu_0 \epsilon_0}} = n_1 \quad \text{for } \mu_1 = \mu_0$$

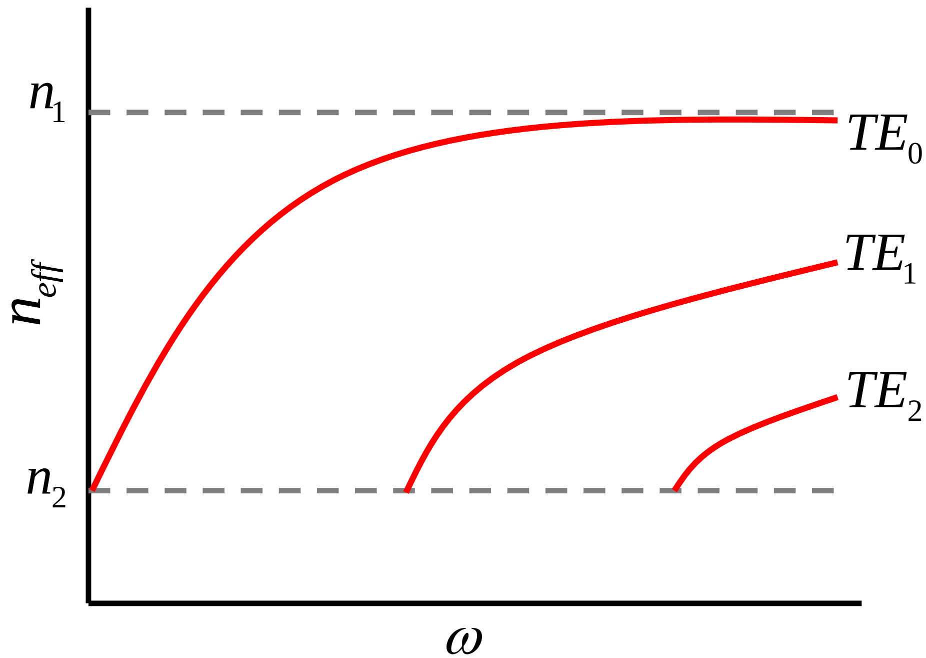
Low-frequency limit

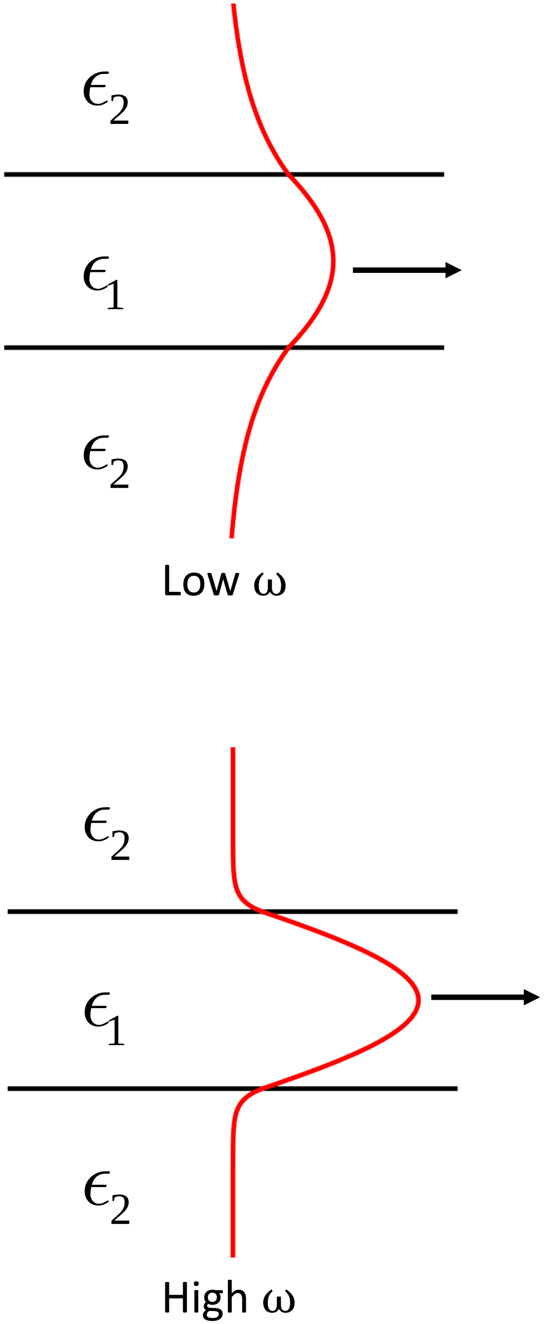
$$\text{Radius} \rightarrow 0 \;\; \text{as } \omega \rightarrow 0$$

$$\therefore \alpha \rightarrow 0$$

$$\begin{aligned} \beta_z^2 &= \omega^2 \mu_2 \epsilon_2 + \alpha^2 \\ &\simeq \omega^2 \mu_2 \epsilon_2 \end{aligned} \qquad n_{eff} = \frac{\beta_z}{\beta_0} = \frac{\sqrt{\mu_2 \epsilon_2}}{\sqrt{\mu_0 \epsilon_0}} = n_2 \quad \text{for } \mu_2 = \mu_0$$

Effective index





Effective index is a measure of how confined the mode is to the core

Optical confinement factor

$$\Gamma = \frac{\text{Power in core}}{\text{Total power in mode}} = \frac{\frac{1}{2}\int_{core} \mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{z}dx}{\frac{1}{2}\int_{total} \mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{z}dx}$$

Weak guidance limit (mode is mostly within cladding)

$$\Gamma \simeq 2\left(\frac{\pi d}{\lambda_0}\right)^2\left(n_1^2 - n_2^2\right)$$

For largest possible *Γ*

(1) Thick core
(2) Small wavelength
(3) Large index contrast

TM modes

$$\mathbf{H} = \hat{y} H_y(x,z) \rightarrow H_y(x,z) = f(x)h(z)$$

Along the z-direction we expect a traveling wave solution

$$h(z) = C_1 e^{-i\beta_z z} + C_2 e^{i\beta_z z}$$

Along the x-direction, we expect a standing wave solution in the waveguide core and evanescent solution in the cladding.

$$f(x) = \begin{cases} A_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ A_2 \cos(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Even solution

$$f(x) = \begin{cases} B_1 e^{-\alpha(|x|-d/2)} & |x| > d/2 \\ B_2 \sin(\beta_x x) & -d/2 \le x \le d/2 \end{cases}$$

Odd solution

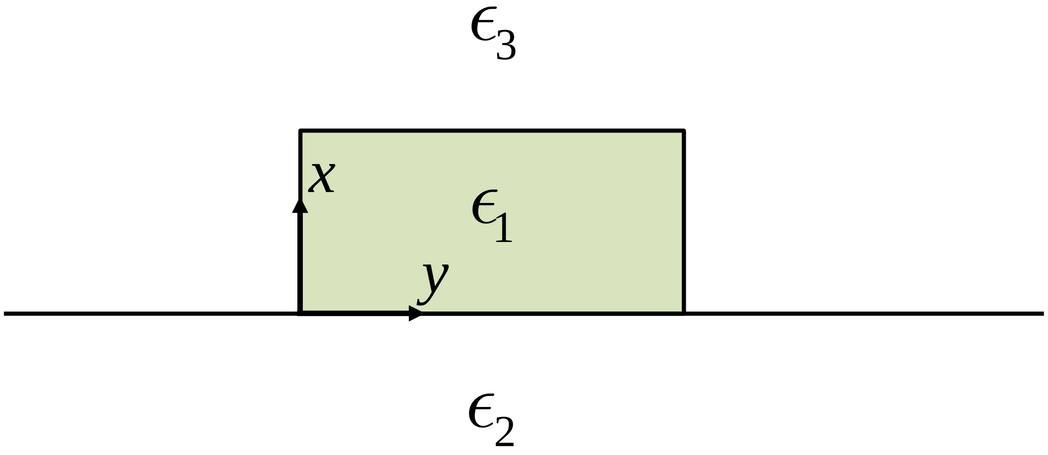
Eigenequations: $$\left(\beta_x \frac{d}{2}\right)^2 + \left(\alpha \frac{d}{2}\right)^2 = \omega^2 \left(\mu_2\epsilon_2 - \mu_1\epsilon_1\right)\left(\frac{d}{2}\right)^2$$

$$\left(\alpha \frac{d}{2}\right) = \frac{\epsilon_2}{\epsilon_1}\left(\beta_x \frac{d}{2}\right)\tan\left(\beta_x \frac{d}{2}\right)$$

$$\left(\alpha \frac{d}{2}\right) = -\frac{\epsilon_2}{\epsilon_1}\left(\beta_x \frac{d}{2}\right)\cot\left(B_x \frac{d}{2}\right)$$

Rectangular waveguides

Rectangular waveguides have dielectric contrast in two-directions



Rectangular waveguides do not support pure TE or TM modes!
Instead they support hybrid modes.

Rectangular waveguides

Hybrid modes

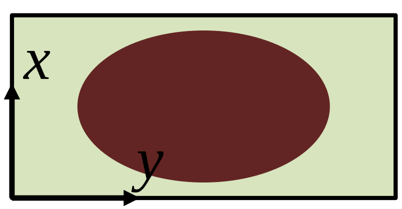
HE_{pq} H_x, E_y are the dominant components (quasi-TE)

EH_{pq} E_x, H_y are the dominant components (quasi-TM)

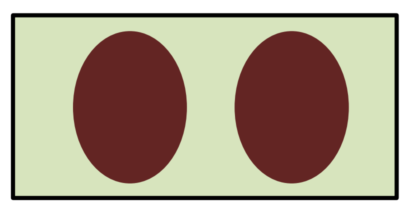
Intensity patterns

$p \rightarrow$ number of nodes in the x-direction

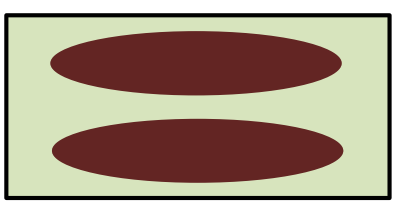
$q \rightarrow$ number of nodes in the y-direction



HE_{00} or EH_{00}

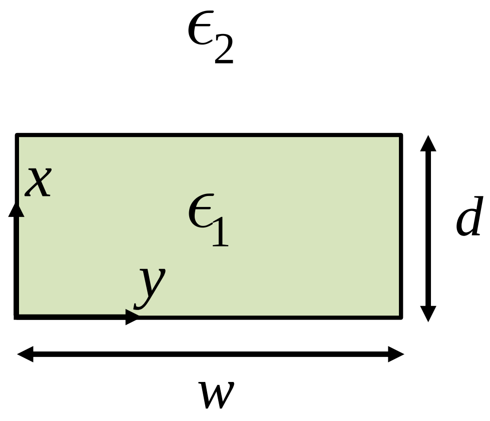


HE_{01} or EH_{01}



HE_{10} or EH_{10}

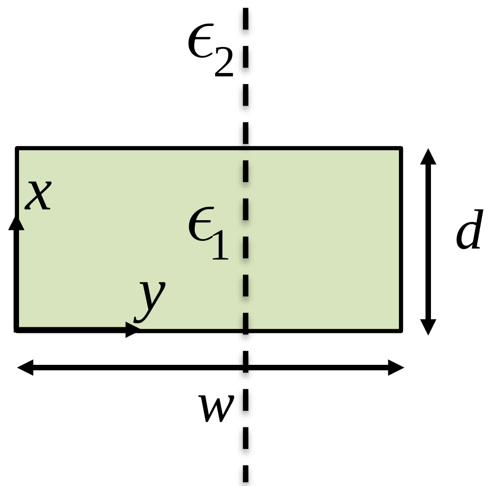
Effective index method



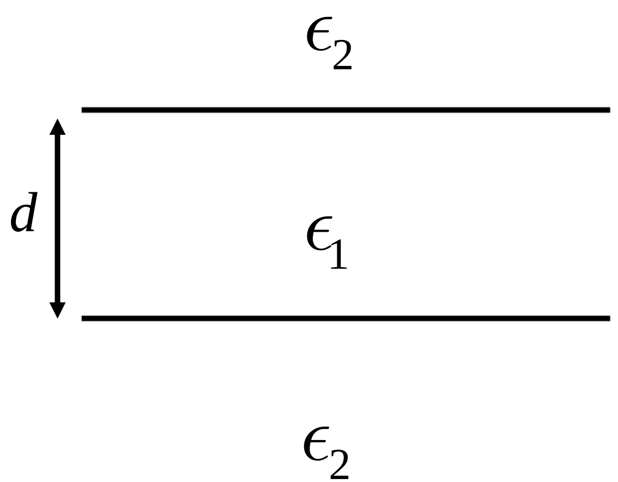
We estimate the propagation constant of the HE_{00} mode with the effective index method. We essentially break the 2D problem into a 1D slab waveguide problem.

To simplify this problem we assume that the waveguide is completely surrounded by the same index. More sophisticated examples are found in the book.

Step 1

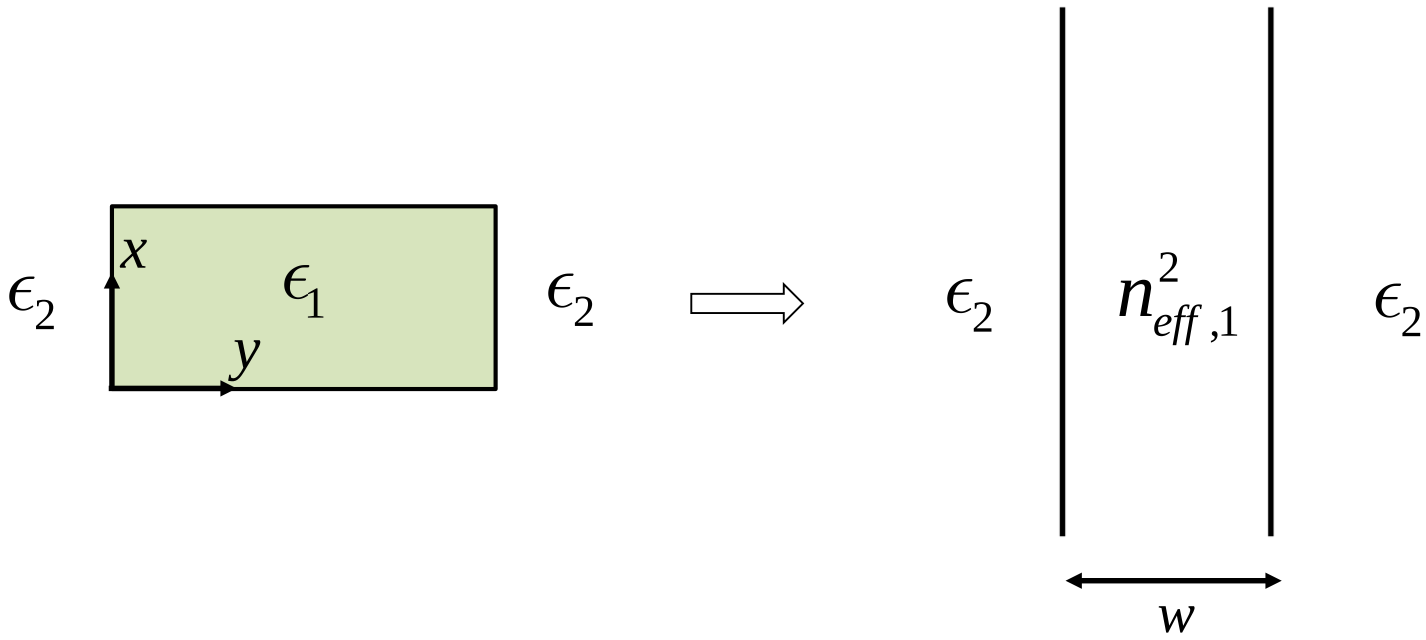


Solve for the **TE mode** of the slab waveguide with core of permittivity ϵ_1 and cladding with permittivity ϵ_2



Calculate the effective index $n_{eff,1}$ and modal distribution $F(x)$

Step 2



Solve for the **TM mode** slab waveguide with core of permittivity n_{eff}^2 and cladding with permittivity ϵ_2. Calculate the propagation constant β_z and modal distribution $\mathrm{G}(y)$.

The overall propagation constant of the 2D waveguide is then β_z and the modal distribution of the 2D waveguide is given by

$$E_y(x, y) = F(x)G(y)$$